A stochastic hierarchical model of city size distribution applied in the Tuscany region

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Abstract We demonstrate how a system of cities self-organizes in a a hierarchical structure which grows with a bottom-up mechanism, so that the resulting distribution is power law. This result is achieved first showing that the power law distribution is the only which fits the balance between demand and offer. Later a stochastic dynamic model is proposed whose numerical results confirm the analytical conclusions, and it is experimented in the analysis and simulation of the urban system of the Tuscany region (Italy).

1 Introduction

One of the most accepted characteristic of an urban system is the regularity of the the city size distribution. This distribution is reasonably well approximated by a power law distribution, where the exponent of the related cumulative distribution varies around the value of -1. Surveys exist such as [13] [31] [35] which report on the city size distribution in different countries, but also a meta-analysis combining estimates from different studies [27] which, in essence, confirm the existence of this characteristic.

The stability of the distribution both in time and space, is considered like a mystery [24], so that several are the explanations of this phenomenon, embedded in different theories. In the identification of the main streams of these theories, opinions are different, beginning from the review of Berry and Garrison [7], in which the Zipf's explanation based on unification and diversification forces, hierarchical approach (Christaller's Central place theory), the Ravshesky's migration theory, and the Simon's preferential attachment are presented as the main leading theories. Later Richardson [30] included in the list of explanations, the hierarchical approach of Beckmann, the allometric growth model, as well as the random growth model, concluding that a satisfactory explanation relies in the consideration of both systematic and stochastic factors. Recently Davis and Weinstein ([15]) consider increasing returns, random growth and location fundamentals as the leading theories in the study of the distribution of activities within a country. Finally, Schaefer [32] considers internal i.e. supply oriented and external i.e. demand oriented approaches, these last being related to the hierarchical structure of a system of cities. Aware of all the mentioned approaches, for the purposes of this study we distinguish between the explanation relying on pure economic principles, and that on the stochastic generative models of power law distribution (a review of that is in [17] and [26]).

Among economic approaches we distinguish a classical one arising with the Central Place Theory [14] and that further developed along with the neoclassical approach first, and by using the monopolistic competition model, later. The classical approach of the Central place Theory is based on the relationship between the center and its market area, replicated at different scale, and resulting in a hierarchical organization. Even if the network-based organization of urban center may coexist with the hierarchical structure [12], urban systems have been usually characterized with their nested hierarchy [22]. The analogy between a cities system and a river network has been indeed pinpoint in [36]. What the Christaller's Central Place Theory, and more explicitly the Beckman's[6], [5], model do (for a review see [4], and [29]), is to build a hierarchy in which at each level a balance exists between the population of the city and that of its basin of attraction including the population of the city itself. [10]. Dynamic versions have been proposed in [2] where the evolution has been obtained via the utilization of the non-linear dynamic emerging from logistic growth, and in [11], where the dynamic results from a complex interplay of hierarchical organization and innovation.

In the classical Central place theory approach, the power law distribution is the outcome of the growth of a hierarchical structure in which the number of branches grows exponentially while the size of each branch decreases exponentially (see [6]). A random factor is usually included in order to transform the step-likewise hierarchy in a continuous one, as suggested by empirical analysis.

Later, beginning from the work of Henderson [21] the focus has changed from the relation city-market area, to the balance between local external economies of scale and dis-economies of urban crowding or congestion costs (see [1] for a synthesis of this approach). This approach based on the opposition of centripetal and centrifugal forces was indeed invoked by many other authors. In this neoclassical approach, agglomeration is due to the external economies while congestion costs arise from the necessity to commute to the central business district and from land rent. External economies are related to the industrial sector in which the city specializes while dis-economies depend on city size, so that a city reaches an optimum size in relation to its specialization, admitting a variability of the the optimum size for each industrial sector.

Finally, based on the monopolistic competition theory, theories and models have been developed in which agglomeration forces result from the interplay of economies of scale, transportation costs and factors mobility [23]. In the strand of this literature known as "New economic geography", the first reason for agglomeration is the existence of economies of scale in production, at the level of the plant and at a higher level of a complex of interrelated activities. What prevents the existence of an only city in which the whole economic activities concentrate, are, from one side, the transportation costs, and on the other side, the dispersion of the agricultural population. The economic system is considered as the result of a self-organized process [25]. Therefore a cumulative process is supposed as the origin of the urban concentration even in absence of natural endowment. In fact, assuming the mobility of production's factors, even a small initial diversity may give rise to a differentiation in size. From this hypothesis the basic core-periphery model arises in which a dispersed agricultural production is supposed as the initial demand for manufactured goods. So that the emergence of a hierarchical urban system is shown when an economy contains industries that differ in terms of scale economies and/or transport costs([18]).

However such deterministic approaches don't result in a power law distribution of cities, unless to suppose an external factor such as the power law distribution of natural advantages[24]. It is for this reasons that generative models of power law distribution such as that based on random growth ([9]) or preferential attachment ([34]) are utilized to explain the power law distribution of cities. The preferential attachment model is utilized to explain the evolution of the urbanization of the new world ([24]) or of to the industrial dynamic ([3]). In this last case cities are clusters of firms so that the power law distribution arises from a similar distribution of firms. These methods, in fact, emphasize the role of the city in the creation of the condition for its growth. However the independence of random growth on size, makes difficult to maintain the hypothesis of increasing returns. Thereby specific economic explanations are proposed which consider economic random shocks not directly connected with size and scale economies. Gabaix explains this aspect in terms of amenities shock (see [19] [20]), Duranton in terms of innovation shocks: cities grow, or decline, as they win, or lose, innovative firms. Therefore, small innovation-driven technological shocks are the main engine behind the growth and decline of cities [16].

The above mentioned theories and models don't attain the power law distribution as a stable state, unless a fine tuning of parameter [16]. In turn, a power law distribution is usually obtained with the utilization of the generative models which are not economic in essence. What we propose in this paper is a theory and a model based on economic arguments and able to result in a power law distribution as a stable state. From the classical approach we take the idea of hierarchy (see Schaefer [32] for the importance to integrate hierarchy and agglomeration forces), while from the new economic geography we borrow the dynamic and self-organized aspect of the economic system [25]. In addition the proposed theory relies on the method of generative models of power law for its stochastic approach. In conclusion the present study combines the hierarchical approach based in the relation among the center and its tributary area, with a stochastic method.

The arguments run as follows. First we summarize a previous model [33] in which it is shown that the power law distribution is the only which fits the balance between demand and offer. Later using the findings of this study, a stochastic dynamic model is proposed and experimented in the analysis and simulation of the urban system of the Tuscany region (Italy).

2 A model based on an hierarchical structure resulting in a power law distribution

In this model (for a detailed explanation see [33]) goods and services produced in a city are distinguished in two types: that which are related with the quantity of inhabitants of the city and that which are not. To the first type belong goods and services that are preferentially available only in cities with a population above an established threshold (see [8]). For goods and services belonging to the first type, exchanges are asymmetric: inhabitants of small towns purchase goods from larger cities [28], but the inverse does not happen, because these goods are not available in small towns. The result is that each city is characterized by its basin of attraction, or market area.

These second type goods result from the specialization of the city. In this case, the concerned demand is not subject to the asymmetric character of the previous demand, in the sense that the inhabitants of a large city may get these second type goods from a small town and vice versa. The result is like the demand for these products comes into the city from everywhere. Hence, in the short period, this demand is considered as exogenous.

We consider a set of cities with a variable population p, where $p \geq p_{\min}$ ($p_{\min}$ being the number of inhabitants in the smallest town). The proposed model is built under the hypothesis of a balance between demand and offer. Demand corresponds to the number of inhabitants, while offer to that of workers. Therefore each city collects the demand coming from the other cities, and from the outside, and offers a proportional amount of commodities. In each city, workers produce goods and services for the demand: (1) coming from smaller cities belonging to the basin of attraction of the city and asking for first type goods (p_d), (2) coming from everywhere and asking for second type goods (p_o), and (3) coming from the population of the same city (p).

While p_o, being independent on p, is established exogenously, p_d needs to be calculated. In essence the key hypothesis runs as follows. In each city the offered variety of commodities is positively correlated with the population, so that the inhabitants of a town demand the goods (not offered in the town where they live) from a larger town or city, and choose this city at random. Under this hypothesis, the population p_d which demands goods and services, to the city with p inhabitants, results as follows:

$$p_d = \int_{p_{\min}}^{p} \frac{q f(q)}{\int_q^\infty f(x)} \mathrm{d}q, \tag{1}$$

where $f(q)$ is the number of cities with q inhabitants.

In each city inhabitants work to satisfy the demand coming from the basin of attraction. Applying the economic base theory we find that:

$$p = \alpha p_d + \beta p_o, \tag{2}$$

where α is the factor connecting the population of the basin of attraction (the demand) with the population of the city (the supply). We substitute equation (1) in equation (2), and we look for the distribution satisfying the resulting equation:

$$p = \alpha \int_{p_{\min}}^{p} \frac{q f(q)}{\int_q^\infty f(x)} \mathrm{d}q + \beta p_o \tag{3}$$

whose solution is the following:

$$f(p) = \frac{c}{\alpha} p^{-\frac{1+\alpha}{\alpha}}. \tag{4}$$

In addition, it results that $p_o = p_{\min}$, and the population of the city is related to the minimum population, so that an increase of this parameter produces a similar increase of the population. This aspect will be utilized in the following dynamic model.

Recalling that r_p, the rank of the city p populated, is $r_p = \int_p^\infty f(x)$, from equation (4) it results that α is the exponent in Zipf's law:

$$p \propto r_p^{-\alpha}. \tag{5}$$

According to the previous expression, the exponent α — usually considered as a proxy for the hierarchical aspect of the urban system — represents the multiplier relating the population of the central city to the population of its basin of attraction.

Having shown how a power law distribution, and a hierarchical structure arise from the balance between offer and demand, in the following section, we include in the model, both spatial and dynamical aspects, so that it can be applied to the urban system of the Tuscany region.

3 The analysis and simulation of the urban system of the Tuscany region

Tuscany is a region in Italy, with a surface of 22 992 squared km, populated on 2001, by about 3.5 millions inhabitants. Main cities are: Florence (356 000 inhabitants), Prato (172 000), and Leghorn (167 000). In total cities and towns are 287 which are considered as autonomous local administrative body (figure 1). The region is delimited from the Tyrrenian sea, west side, and from the Apennines mountain in the north and east sides. In addition the region is an interconnected area having been, unless a limited part in the norther zone, an autonomous state during about three centuries till to 1859.

The administrative borders of the town administration are usually able to contain the whole town expansion, with the exception of some important towns such as the chief town, Florence, where the expansion of the city has overcame these borders. In addition, as usual for cities under an high rent pressure, people, and firms relocated in the administrative territory of the bordering towns, in order to find a more affordable rent price. For this reason the actual number of inhabitants, does not account for the important role of the city. In order to overcome this difficulty one may sum up the population of the central area with that of the bordering towns and locate all this virtual population in the central city. Because we wishes to maintain the individuality of the bordering towns we have utilized a different method. We have thus recalculated the population of the chief town in this way. The population of the chief city has been supposed to growth at least at the average growth rate of the region. In case this theoretic value is lower than observed the population of the city is updated to this value and a corresponding number of inhabitants is subtracted to the

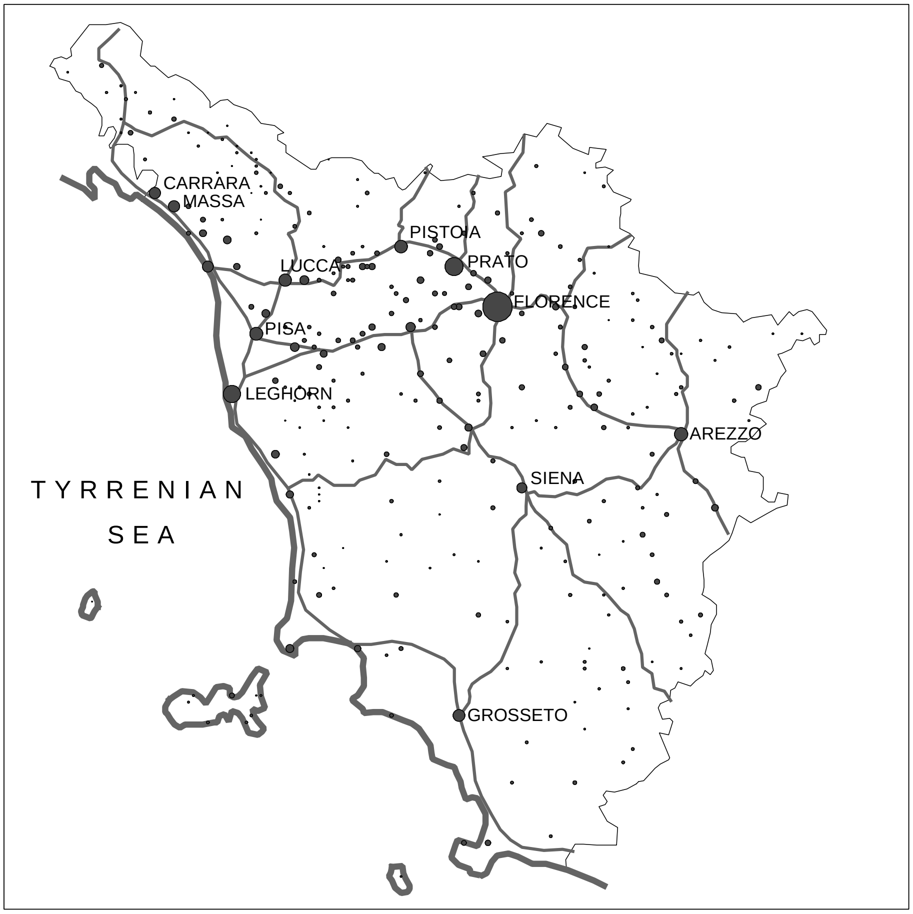

Figure 1: Cities and town, and main roads network in the Tuscany region. The area of circles is proportional to the inhabitants, 2001.

bordering towns in proportion to their population. By using this method it is like we force the outgoing population to come back in the central city. The result is that the population of Florence instead of 356 000 results as 456 000 on 2001, while the difference is subtracted to the bordering towns. This recalculation method has been applied also to the town of Pisa and Carrara, where the situation is similar to that of Florence.

After the recalculation it has been possible to evaluate the distribution of the cities size. As figure 2 shows, the distribution has a power law shape till a cut-off, situated around the size of 7 000. However, not considering towns below the cut-off, in the relation $p_r = p_1 r^{-\alpha}$ we get $p_1 = 363\,690$ and $\alpha = 0.82425$.

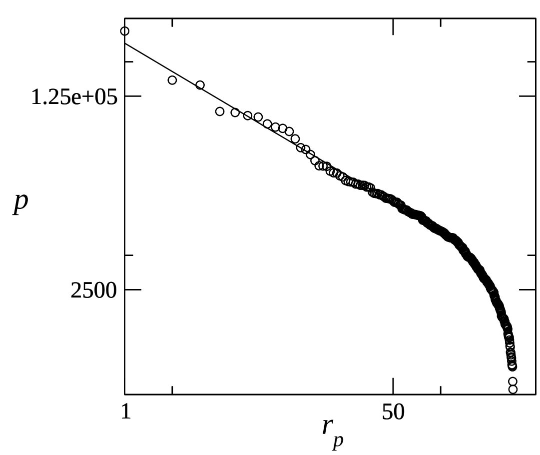

Figure 2: The relation $p \propto r^{-\alpha}$, where $\alpha = 0.83118$.

3.1 The dynamic model

In this section we include in the model, both spatial and dynamical aspects (for a detailed explanation see [33]). Cities are in fact located on a surface. Thereby the space plays a crucial role in the process a city chooses the larger city where to buy the first type goods. In order to choose, the inhabitants minimize the distance from the larger city, which is calculated as in the following expression:

$$d^{\star}_{ij} = d_{ij}(1 + g^d). \tag{6}$$

where d_{ij} is the physical distance, and g^d is a Gaussian random variable with mean equal zero and standard deviation σ^d, which represents a measure of the weight of the spatial aspect in the choice process: the greater the value of σ^d, the lesser the weight of space in the choice's process.

The dynamic spatial model is formulated in the following way. A set of N cities, ($N = 287$ in the case of the Tuscany region) randomly scattered over a surface, each with p_i inhabitants, randomly assigned at the beginning, is established. Hence at each step the inhabitants of a city i choose, among the larger cities, the city j, which minimizes $d^{\star}_{ij}$. The set Ω_j of the cities choosing j is established, and the demand is calculated as the sum of the population of the cities belonging to the set, plus the factor connected with $p_{\min}$. The population of the city in the next time step is updated as in the following equation:

$$p_j(t+1) = p_j(t) + \frac{\alpha \sum_{k\in\Omega_j} p_k(t) + p_{\min}(t)(1 + g'_j) - p_j(t)}{D} \tag{7}$$

where α, as in equation (2), is a proportionality factor applied to the demand, and D is a delay which accounts for the steps necessary in order the offer fits the demand. The parameter g'_j has been established by using the data at the initial step. In fact for each town a value Δ_j is calculated as the difference between the population of the city and the demand addressed to the city and multiplied by α: $\Delta_j = p_j - \alpha \sum_{k\in\Omega_j} p_k(t)$. In other words Δ_j is nothing but the theoretic value of $p_{\min}$ for the considered town j. The distribution of Δ_j can be approximated with a Gaussian distribution as figure 3 shows. So that, the mean $\overline{\Delta_j}$, has been calculated all over the 287 towns and $p_{\min} = \overline{\Delta_j}$. Finally: $g'_j = \frac{(\Delta_j - p_{\min})}{p_{\min}}$.

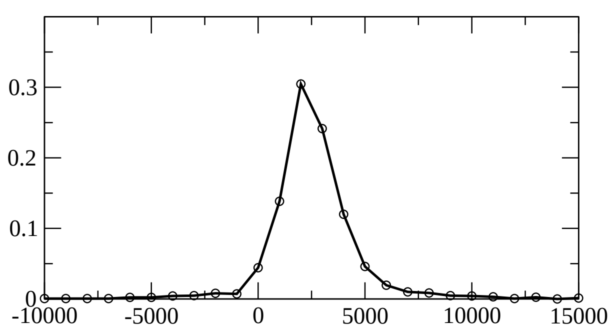

Figure 3: The average distribution of the probability of Δ_j, calculated over 14 census data, related to the period 1861–2001.

3.2 The model applied to the Tuscany region

In order to apply the model to the Tuscany region, the main roads network has been considered in the calculation of distance. In fact, distance is considered as composed of two parts. The first part is that traveled through the network and the second outside the network: from the origin to the initial network's node and from the final network's node to the destination. While the first part utilizes the minimum path method, the second part is calculated as a crow flies and weighted with a parameter. Once calculated the distance between each couple of cities it has been possible to apply the choice method. The resulting spatial configuration is shown in figure 4, while the hierarchical structure, obtained from the spatial configuration, by rearranging the coordinates of towns, is shown in figure 5.

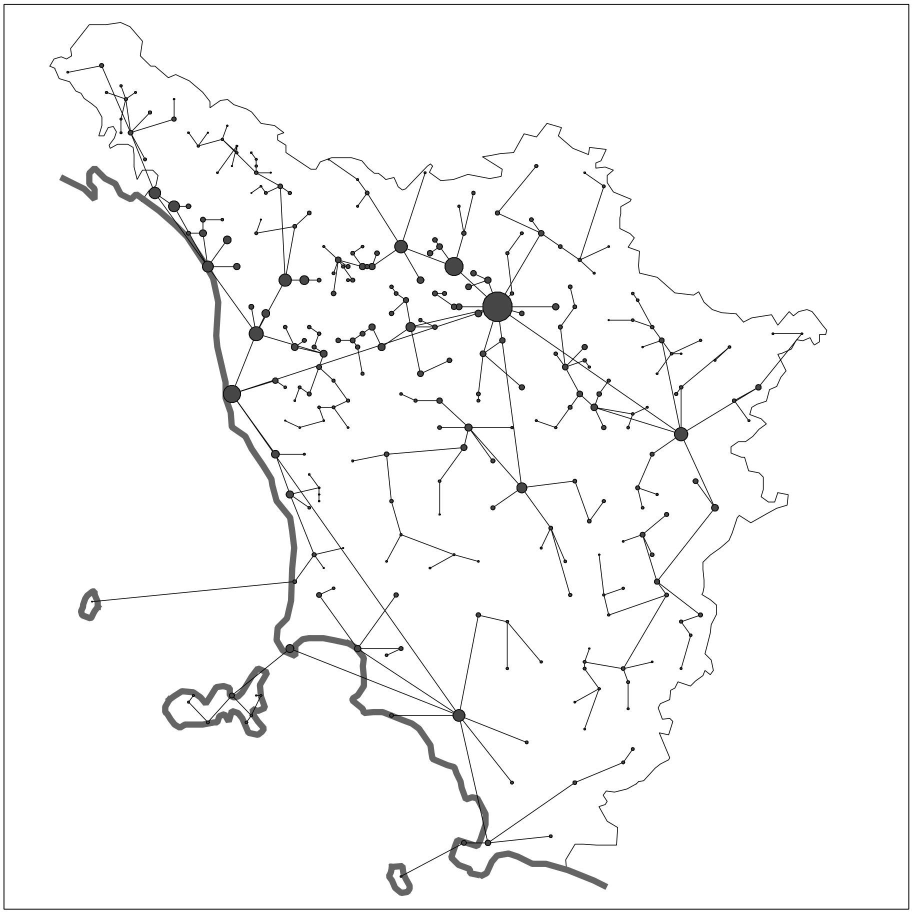

Figure 4: The spatial configuration of the urban system of the Tuscany region.

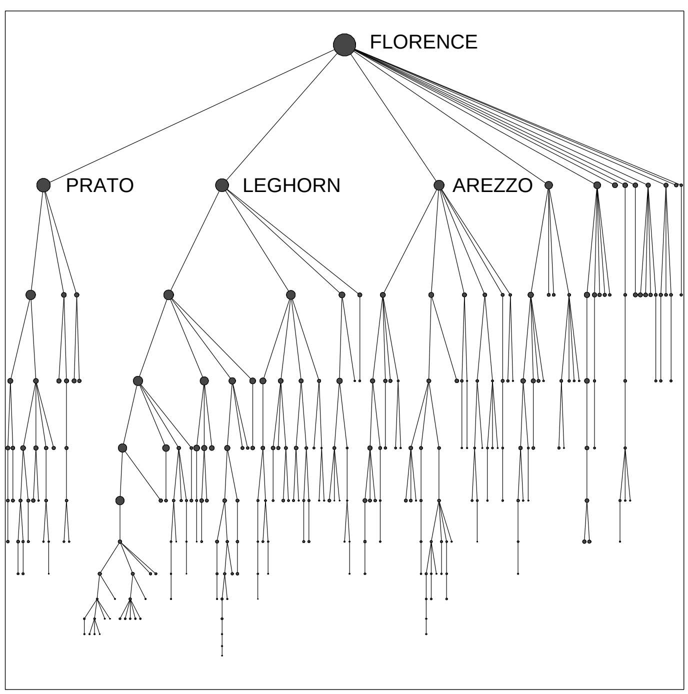

Figure 5: The hierarchical structure of the urban system of the Tuscany region, 2001.

The relation between the population of the city and that of the sum of the linked cities is shown in fig 6. By using the statistical regression method we get: $p = 4507.6 + 0.73088 d_p$ so that $p_{\min} = 4507.6$ and $\alpha = 0.73088$.

Because $p_{\min}$ plays a crucial role in the growth of the system, the relation between $p_{\min}$ and P_T has been analyzed. The value of $p_{\min}$ has been calculated with the previously explained method over a period ranging from 1861 to 2001 in which data are available each ten years unless 1891. The plotted result, see figure 7, shows a clear linear relation between the two variables. When all the periods are considered the result is: $P_T = 496\ 705 + 708.86 p_{\min}$, meaning that an important part of the population (i.e. the constant term) was living in a self-sufficiency economy, such as that based on traditional agriculture. The result changes when only the period 1961-2001 is considered. In this case the result is: $P_T = 26871 + 813.93 p_{\min}$, where the constant is dramatically decreased.

The linearity of the relation $P_T = f(p_{\min})$, makes possible the simulation of the growth of the system. In fact the model runs from the initial population on 1961. The growth is obtained by increasing the value of $p_{\min}$, so that $p_{\min}(t+1) = (1+\gamma) p_{\min}(t)$, where γ is an established growth rate. The model runs till the total population of the region on 2001, is reached. During this period the population of the region grows from 3.28 millions to 3.5 millions inhabitants. In figure 8 a comparison is shown between the simulated and observed inhabitant at the end of the simulation as well as of the two cumulative distributions.

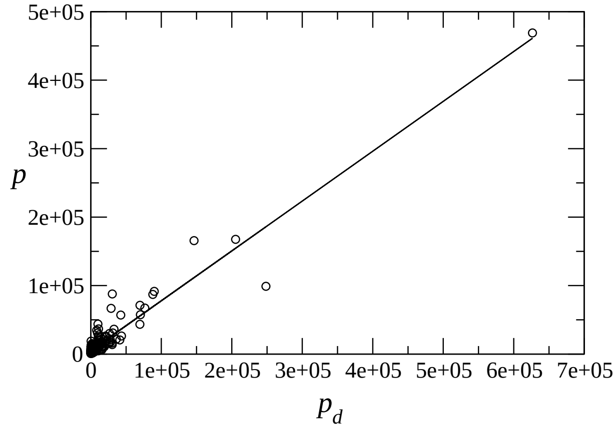

Figure 6: The estimation of the relation $p = \alpha p_d$, where $\alpha = 0.73088$.

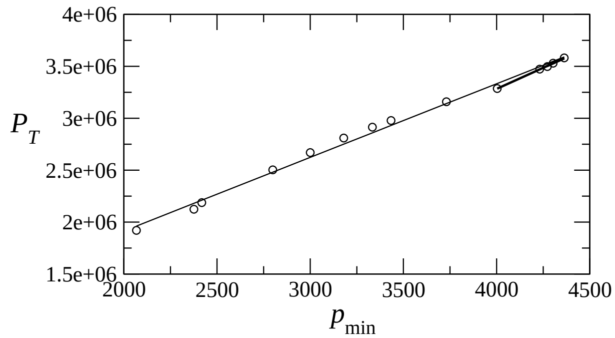

Figure 7: The relation $p_{\min}$, P_T. The tin line represent the regression calculated all over the observations, while the thick line represents the regression calculated in the period 1961-2001

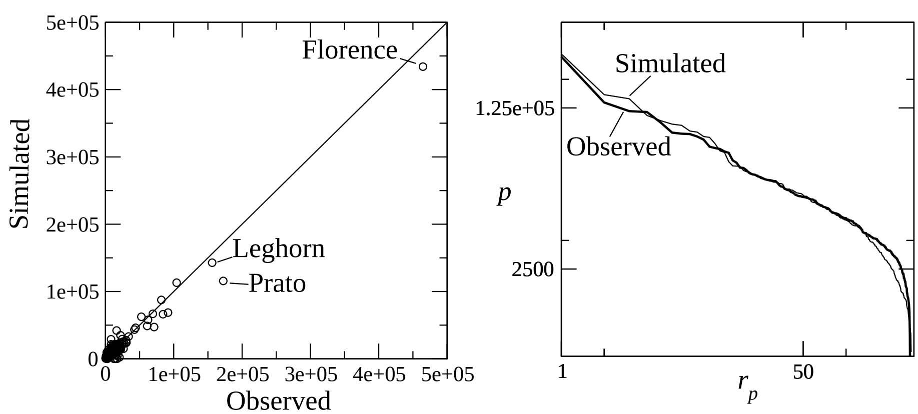

Figure 8: Left: the comparison between the observed and simulated values of the population of the town in the final step of the simulation. Right: the comparison between the observed (tick line) and simulated (tin line) cumulative distribution.

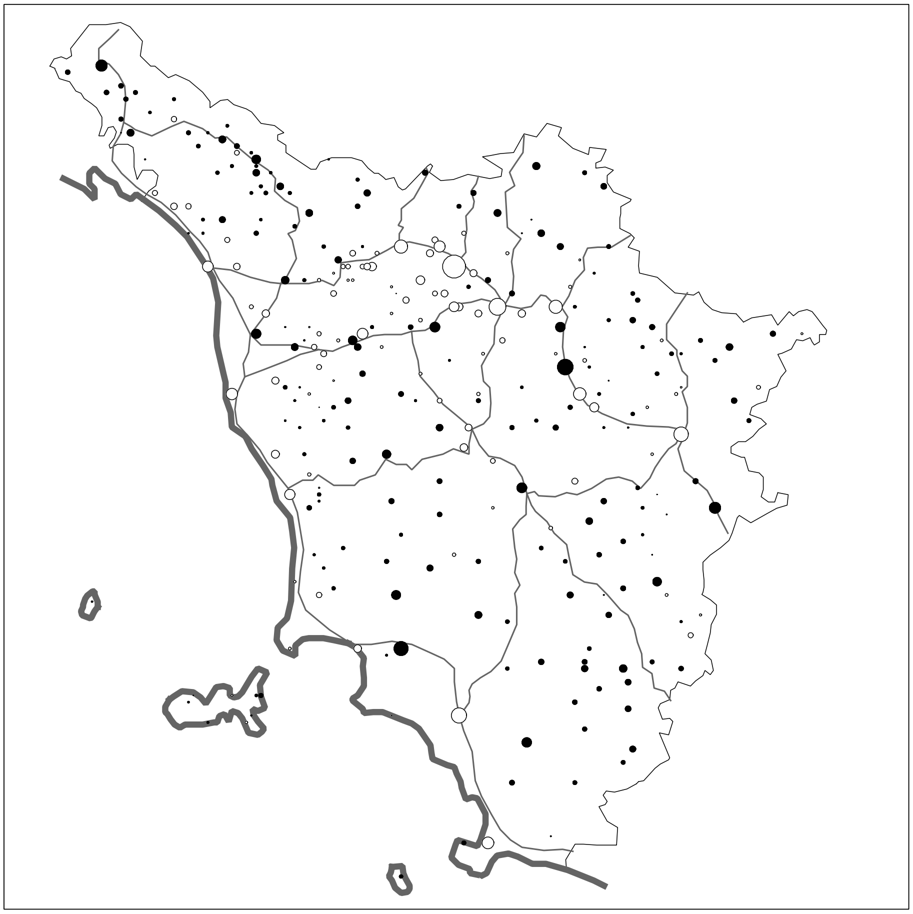

Figure 9: Differences: calculated minus observed inhabitants at the end of the simulation. White circles: negative differences (minimum value: −56 822) black circles positive differences (maximum value: 25 668).

The difference between the simulated and calculated inhabitants for each city and town is shown in figure 9.

Finally we propose the utilization of the model for building scenarios, in relation to the future foreseen changes. One of the reason for the variation of $p_{\min}$ is the demand coming from the outside the region and addressed in various way to the different towns whose economic growth can be partially explained with the economic base theory. For this reason the relation between the value of Δ_i and the value of economic base E_i calculated for each town by using the location quotient method in relation only to the industrial sectors, is shown in figure 10. By using the classical what-if approach we simulate the possible decreasing in the following years of the light industry, namely textile and leather-wear which has been responsible in the recent past of and important share of the economic growth of the region, and now is facing the strong economic competition coming from Asiatic countries. The employees in the textile and leather-wear sectors

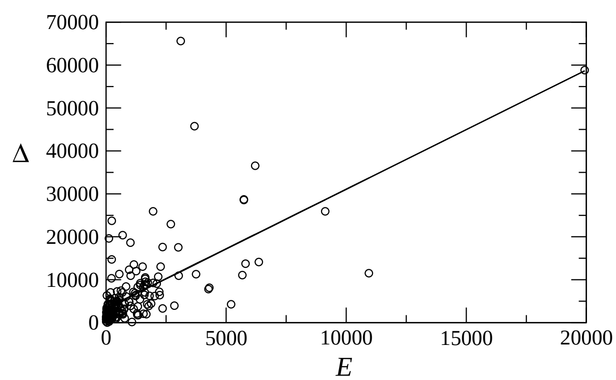

Figure 10: . The relation between the economic base E_i, and the value of Δ_i

Figure 11: Differences resulting from the scenario in which the decreasing of the employed in light industry has been included. White circles: negative differences from a simulation without decreasing of light industry (minimum value: −69 753) black circles positive differences (maximum value: 13 458).

have been subtracted to the variation of $p_{\min}$ (figure 11 shows the result of the simulation). Due to the interconnected non-linear dynamics of the model, the simulation results in a decrease of the population of towns in which light industry is concentrated (see figure 12 for comparison) as well as of towns depending on the demand of these towns, such as Florence.

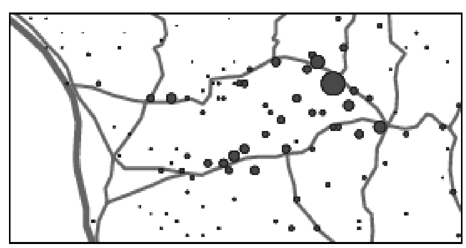

Figure 12: The employed in light industry in 1991 in the core of the region.

4 Conclusions

In this paper it has been shown how a hierarchy arising from a bottom-up mechanism may explain the power law distribution of cities. This bottom-up choice depends on the asymmetrical exchange among cities, based on the implicit variation of the offer as a function of the city size. Each city chooses the city greater in size where to buy goods and services not available in the city itself.

The self-organizing character has made possible to apply it to the simulation of the development of the cities and town of Tuscany region. So that the proposed explanation is able to reproduce both the macroscopic and the microscopic aspects of the cities size distribution.

Finally the application of the model to the production of a scenario based in the probable decrease of employees in light industry, has shown how the effects of the hypothesis propagate in a non-linear way throughout the cities system of Tuscany.

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